ELLIPSES.

THE third Bicircloid Curve, produced by two circular movements (in the same plane) when the angular velocities are as $2:1$, in contrary directions, has for its central equations

$$x = \cos \phi + e \cos \phi = (1 + e) \cos \phi;$$
$$y = \sin \phi - e \sin \phi = (1 - e) \sin \phi;$$

which, by putting $a = 1 + e$ the apocentral radius, and $p = 1 - e$ the pericentral radius, become

$$x = a \cos \phi, \text{ and } y = p \sin \phi.$$

Hence $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{p}\right)^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

Therefore the Curve is an Ellipse, of which a and p are the apsidal major and minor semi-axes, respectively.

The excentricity of this Ellipse is

$$a_e = \sqrt{a^2 - p^2} = \sqrt{(a + p)(a - p)} =$$
$$\sqrt{(1 + e + 1 - e)(1 + e - 1 + e)} = \sqrt{2 \times 2e} = 2\sqrt{e}. \quad (2)$$

Or, the excentricity = the distance from the center to either focus, is a mean proportional between the diameters of the Deferent and Epicycle.

Substituting $r \cos \theta$ for x, $r \sin \theta$ for y in (1.), and clearing from fractions, there results

$$p^2 r^2 \cos^2 \theta + a^2 r^2 \sin^2 \theta = a^2 p^2;$$

whence $r^2 = \dfrac{a^2 p^2}{p^2 \cos^2 \theta + a^2 \sin^2 \theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$

Putting in the denominator, $a^2 - a^2 e^2$ for p^2, there results

$$r^2 = \dfrac{a^2 p^2}{a^2 - a^2 e^2 \cos^2 \theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again, putting $\dfrac{1}{2}(1 + \cos 2\theta)$ for $\cos^2 \theta$, in the denominator of (4), there results.

$$r^2 = \dfrac{a^2 p^2}{\left(a^2 - \dfrac{1}{2} a^2 e^2\right) - \dfrac{1}{2} a^2 e^2 \cos 2\theta} \\
= \dfrac{2 a^2 p^2}{a^2 + p^2 - a^2 e^2 \cos 2\theta}. \qquad \Biggr\} \quad . \quad . \quad . \quad (5)$$

In the equations (1), (3), (4), and (5), the center is the origin or pole.

The polar equation when the focus is the pole is

$$r = \frac{p^2}{a + a \cdot \cos\theta}. \qquad \cdots \cdots \cdots (6)$$

When the Epicyclic radius vanishes, or $e = 0$, then $a = p$; in this case equation (1) becomes

$$x^2 + y^2 = a^2 = p^2; \qquad \cdots \cdots \cdots (7)$$

or

$$r = a = p. \qquad \cdots \cdots \cdots \cdots (8)$$

Hence in this case the curve is a circle, whose radius is a or p.

Similarly, equation (3) gives, as in (8),

$$r^2 = \frac{a^4}{a^2 \cos^2\theta + a^2 \sin^2\theta} = a^2; \text{ or } r = a = p.$$

When the Epicyclic radius becomes equal to the Deferent radius, or $e = 1$, then $a = 2$, $p = 0$; and equation (1) becomes

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{0}\right)^2 = 1; \text{ or } y = 0.$$

The curve in this case becomes a straight line, coinciding with the axis of x.

Equation (3) gives $r = 0$, since $p = 0$; but here must be excepted the case of $\sin\theta = 0$, in which case equation (3) becomes $r^2 = \frac{a^2 p^2}{p^2} = a^2$; $\therefore r = \pm a$; and these are the limiting values of r.

The polar equation of the third Bicircloid, when centric, gives $r = a\cos\frac{2\theta}{0}$; which, when $\theta = 0$, or $\theta = 180°$, gives, as before, $r = +a$, or $r = -a$, the limiting values.

This curve is the *Rectloid* or *Orthoid*; being a finite straight line, whose length is equal to the Apocentral Diameter.

The values $e = 0$, and $e = 1$, give the two extreme cases of the curve.

If we suppose e infinite, then $a = c$, $p = -e$; and we get, both at once, the two cases in one; for the curve becomes a circle: while, its radius being infinite, the circle is equivalent to a straight line: but here the length is not finite, because it is not centric. This, however, is an Imaginary Bicircloid.

Let $D = 1$, $e = \frac{m}{n}$; then

$$p = \frac{n-m}{n}; \qquad a = \frac{n+m}{n}. \quad \ldots \ldots \quad (9)$$

$a\epsilon = 2\sqrt{e}$, by (2) $\therefore a\epsilon = \frac{2\sqrt{mn}}{n}$; $\epsilon = \frac{2\sqrt{mn}}{n+m}$. $\ldots$ (10)

Let $a = 1$, $p = \frac{m}{n}$, then

$$D = \frac{n+m}{2n}; \quad E = \frac{n-m}{2n}; \quad a\epsilon = \epsilon = \frac{\sqrt{n^2-m^2}}{n}. \ldots \quad (11)$$

Let $a = 1$, $\epsilon = \frac{m}{n}$, then $p = \frac{\sqrt{n^2-m^2}}{n}$. $\ldots \ldots$ (12)

$$D = \frac{n+\sqrt{n^2-m^2}}{2n}; \quad E = \frac{n-\sqrt{n^2-m^2}}{2n}. \quad \ldots \ldots \quad (13)$$

The letter v, c, or f, attached as an index to the letters x and r, denotes that the origin, or the pole, is at the *vertex, center,* or *focus,* respectively.

Thus, when the center is the origin, we have $a^2 y^2 + p^2 x_c^2 = a^2 p^2$; from equation (1).

Whence $a^2 y^2 + p^2 = a^2 - x_c^2$; which is the second form of the rectangular equation.

If we make the vertex the origin, put $x_v + x_c = a$; then $x_c = a - x_v$; and substituting this in equation (1), we have $a^2 y^2 + p^2 = a^2 - (a - x_v)^2 = 2 a x_v - x_v^2$; as in the first form.

When the origin is at the focus, $x_f + x_c = a\epsilon$; $\therefore x_c = a\epsilon - x_f$; and $\quad a^2 y^2 + p^2 = a^2 - (a\epsilon - x_f)^2 = a^2 - a^2\epsilon^2 + 2 a\epsilon x_f - x_f^2$
$$= p^2 + 2\sqrt{a^2 - p^2}.\, x_f - x_f^2;$$

which is the third form.

The polar equations are deduced from the rectangular equations by making

$$x_v = r_v \cos\theta_v, \quad y = r_v \sin\theta_v;$$
$$x_c = r_c \cos\theta_c, \quad y = r_c \sin\theta_c;$$
$$x_f = r_f \cos\theta_f, \quad y = r_f \cos\theta_f;$$

according as the pole is at the vertex, at the center, or at the focus, respectively.

Thus, we have

$$a^2 y^2 + p^2 = 2 a x_v - x_v^2;$$

whence

$$a^2 r_v^2 \sin^2 \theta_v + p^2 = 2 a r_v \cos \theta_v - r_v^2 \cos^2 \theta_v,$$

or

$$a^2 r_v \sin^2 \theta_v = 2 a p^2 \cos \theta_v - p^2 r_v \cos^2 \theta_v;$$

$$\therefore r_v = \frac{2 a p^2 \cos \theta_v}{a^2 \sin^2 \theta_v + p^2 \cos^2 \theta_v} = \frac{2 a p^2 \cos \theta_v}{a^2 - (a^2 - p^2) \cos^2 \theta_v};$$

and putting $\dfrac{1 + \cos 2 \theta_v}{2}$ for $\cos^2 \theta_v$, we have

$$r_v = \frac{2 a p^2 \cos \theta_v}{a^2 - \dfrac{a^2 - p^2}{2} - \dfrac{a^2 - p^2}{2} \cos 2 \theta_v} =$$

$$\frac{4 a p^2 \cos \theta_v}{2 a^2 - a^2 + p^2 - (a^2 - p^2) \cos 2 \theta_v} = \frac{2 \times 2 a p^2 \cos \theta_v}{a^2 + p^2 - (a^2 - p^2) \cos 2 \theta_v}.$$

If we put $\epsilon \cos \theta = \cos \psi$, then

$$a^2 \epsilon^2 \cos^2 \theta = (a^2 - p^2) \cos^2 \theta = a^2 \cos^2 \psi;$$

and we have also

$$r_v = \frac{2 a p^2 \cos \theta_v}{a^2 - a^2 \cos^2 \psi_v} = \frac{2 a p^2 \cos \theta_v}{a^2 \sin^2 \psi_v} = \frac{2 p^2 \cos \theta_v}{a \sin^2 \psi_v}.$$

Since $a^2 \epsilon^2 = a^2 - p^2$, equation (4) may be written

$$r_c^2 = \frac{a^2 p^2}{a^2 - (a^2 - p^2) \cos^2 \theta_c};$$

and putting $a^2 \cos^2 \psi$ for $(a^2 - p^2) \cos^2 \theta_c$, we have

$$r_c^2 = \frac{a^2 p^2}{a^2 - a^2 \cos^2 \psi_c} = \frac{a^2 p^2}{a^2 \sin^2 \psi_c} = \frac{p^2}{\sin^2 \psi_c}; \quad \therefore r_c = \frac{p}{\sin \psi_c}.$$

Similarly, equation (6) may be written

$$r_f = \frac{p^2}{a + \sqrt{a^2 - p^2} \cos \theta_f} = \frac{p^2}{a + a \cos \psi_f} = \frac{p^2}{a(1 + \cos \psi_f)} = \frac{p^2}{2 a \cos^2 \dfrac{\psi_f}{2}}.$$

The angle ψ may be geometrically constructed as follows: On the major axis of the ellipse, as a diameter, describe a semicircle. Project orthogonally on the radius vector, or its prolongation, the distance between the foci, viz. $2 a \epsilon$. At the extremity of the major axis inscribe a chord in the semicircle, which shall be equal to the preceding projection of $2 a \epsilon$; this chord makes with the major axis an angle $= \psi$.

If ϖ denote the perpendicular on the tangent, then according as the pole is taken at the *vertex, focus,* or *center,* so we have

$$\varpi_v = \frac{p}{a\,e}\;\frac{\sqrt{p^4 + a^2 e^2 r_v^2} - p^2}{\sqrt{2\sqrt{p^4 + a^2 e^2 r_v^2} - p^2(2 - e^2) - e^2 r_v^2}}.$$

$$\varpi_f = p\sqrt{\frac{r_f}{2\,a - r_f}}.$$

$$\varpi_c = \frac{a\,p}{\sqrt{a^2 + p^2 - r^2}}.$$

If x' and y' are the coordinates of a point in the curve which corresponds to the angle $\frac{\pi}{2} - \phi$; x'', y'', those corresponding to the angle $\phi - \frac{\pi}{2}$; and x''', y''', those corresponding to the angle $\phi + \frac{\pi}{2}$; then we have

$$x\,y' + x'\,y = x''\,y - x\,y'' = x\,y''' - x'''\,y = e\,p.$$
$$r^2 + r'^2 = r^2 + r''^2 = r^2 + r'''^2 = a^2 + p^2.$$

The relation between θ_c and ϕ is

$$(1 - e^2 \cos^2 \theta_c) \times \left(1 + \frac{e^2}{1 - e^2} \cos^2 \phi\right) = 1.$$

Whence

$$e \cos \phi \cos \theta_c = \sqrt{(\cos \phi + \sqrt{1 - e^2} \cos \theta_c)(\cos \phi - \sqrt{1 - e^2}\cos \theta_c)};$$

$$\cos \theta_c = \frac{\cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}}, \quad \text{and} \quad \cos \phi = \frac{\sqrt{1 - e^2}\cos \theta_c}{\sqrt{1 - e^2 \cos^2 \theta_c}}.$$

Let u and w be the coordinates of the Evolute, respectively parallel to the coordinates x and y, then

$$u = \frac{4\,e}{1 + e}\cos^3 \phi = \frac{a^2 e^2}{a}\cos^3 \phi, \quad w = -\frac{4\,e}{1 - e}\sin^3 \phi = -\frac{a^2 e^2}{p}\sin^3 \phi,$$

whence

$$(pw)^{\frac{2}{3}} + (a\,u)^{\frac{2}{3}} = (a\,e)^{\frac{4}{3}}.$$

and

$$\sqrt[3]{pw} = \sqrt{(\sqrt[3]{a^2 e^2} + \sqrt[3]{a\,u})(\sqrt[3]{a^2 e^2} - \sqrt[3]{a\,u})}.$$

The following are examples of ELLIPSES whose principal axes are proportioned, to each other, according to the series of ratios expressed by every pair of numbers not exceeding 10.
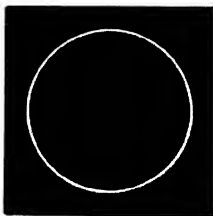

Fig. 1.   Axes as 1 : 1.

Radii as 1 : 0; $e = 0$.

$a = 1, p = 1$; $1^2 e^2 = 1^2 - 1^2 = 0$.

The *foci* at the *center*.

A Circle, one extreme of the Ellipse: the other being the Orthoid, fig. 33.

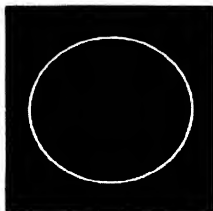$$y^2 = 2 x_e - x_e^2 = 1 - x_e^2 = 1 - x_f^2. \qquad r_e = 2 \cos \theta_e. \qquad r_e = r_f = 1.$$

Fig. 2.   Axes as 10 : 9.

Radii as 19 : 1; $e = 0{\cdot}05263$.

$a = 1, p = 0{\cdot}9$; $10^2 e^2 = 10^2 - 9^2 = 19$.

$$10^2 y^2 \div 9^2 = 2.10 x_e - x_e^2 = 10^2 - x_e^2$$
$$= 9^2 + 2 \sqrt{10^2 - 9^2} . x_f - x_f^2.$$

$r_e = 2.10.9^2 \cos \theta_e \div \{10^2 - (10^2 - 9^2) \cos^2 \theta_e\}$

$\quad = 2.9^2 \cos \theta_e + 10 \sin^2 \psi_e. \quad r_e^2 = 10^2.9^2 + \{10^2 - (10^2 - 9^2) \cos^2 \theta_e\},$

$r_e = 9 + \sin \psi_e. \; r_f = 9^2 + \{10 + \sqrt{10^2 - 9^2} . \cos \theta_f\} = 9^2 + 2.10 \cos^2 \dfrac{\psi}{2}.$
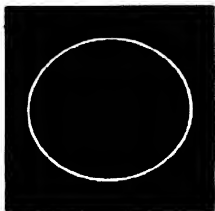
Fig. 3.   Axes as 9 : 8.

Radii as 17 : 1; $e = 0{\cdot}0588$.

$a = 1; \; p = 0{\cdot}8$; $9^2 e^2 = 9^2 - 8^2 = 17$.

$$9^2 y^2 \div 8^2 = 2.9 x_e - x_e^2 = 9^2 - x_e^2$$
$$= 8^2 + 2 \sqrt{9^2 - 8^2} x_f - x_f^2.$$

$r_e = 2.9.8^2 \cos \theta_e + \{9^2 - (9^2 - 8^2) \cos^2 \theta_e\}$

$\quad = 2.8^2 \cos \theta_e + 9 \sin^2 \psi_e. \quad r_e^2 = 9^2.8^2 + \{9^2 - (9^2 - 8^2) \cos^2 \theta_e\},$

$r_e = 8 + \sin \psi_e. \quad r_f = 8^2 + \{9 + \sqrt{9^2 - 8^2} . \cos \theta_f\} = 8^2 + 2.9 \cos^2 \dfrac{\psi_f}{2}.$
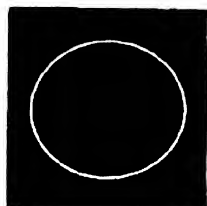
Fig. 4.   Axes as 8 : 7.

Radii as $15 : 1$; $e = 0.06$.

$a = 1$, $p = 0.875$; $8^2 e^2 = 8^2 - 7^2 = 15$.

$8^2 y^2 \div 7^2 = 2.8\, x_p - x_p^2 = 8^2 - x_c^2$

$\qquad\qquad = 7^2 + 2\sqrt{8^2 - 7^2}.\, x_f - x_f^2.$

$r_p = 2.8.7^2 \cos\theta_p \div \{8^2 - (8^2 - 7^2)\cos^2\theta_p\}$

$\quad = 2.7^2\cos\theta_p \div 8\sin^2\psi_p.\quad r_c^2 = 8^2.7^2 \div \{8^2 - (8^2 - 7^2)\cos^2\theta_c\},$

$r_c = 7 \div \sin\psi_c.\quad r_f = 7^2 \div \{8 + \sqrt{8^2 - 7^2}.\cos\theta_f\} = 7^2 \div 2.8\cos^2\dfrac{\psi_f}{2}.$
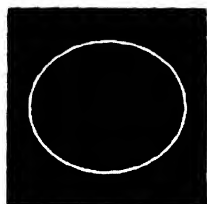
Fig. 5.   Axes as 7 : 6.

Radii as $13 : 1$; $e = 0.0769$.

$a = 1$, $p = 0.857142$; $7^2 e^2 = 7^2 - 6^2 = 13$.

$7^2 y^2 \div 6^2 = 2.7\, x_p - x_p^2 = 7^2 - x_c^2$

$\qquad\qquad = 6^2 + 2\sqrt{7^2 - 6^2}\, x_f - x_f^2.$

$r_p = 2.7.6^2 \cos\theta_p \div \{7^2 - (7^2 - 6^2)\cos^2\theta_p\}$

$\quad = 2.6^2\cos\theta_p \div 7\sin^2\psi_p.\quad r_c^2 = 7^2.6^2 \div \{7^2 - (7^2 - 6^2)\cos^2\theta_c\},$

$r_c = 6 \div \sin\psi_c.\quad r_f = 6^2 \div \{7 + \sqrt{7^2 - 6^2}.\cos\theta_f\} = 6^2 \div 2.7\cos^2\dfrac{\psi_f}{2}.$
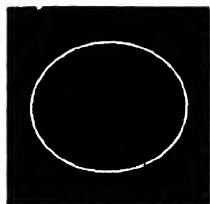

Fig. 6.   Axes as 6 : 5.

Radii as $11 : 1$; $e = 0.09$.

$a = 1$, $p = 0.83\dot{3}$; $6^2 e^2 = 6^2 - 5^2 = 11$.

$6^2 y^2 \div 5^2 = 2.6\, x_p - x_p^2 = 6^2 - x_c^2$

$\qquad\qquad = 5^2 + 2\sqrt{6^2 - 5^2}\, x_f - x_f^2.$

$r_p = 2.6.5^2\cos\theta_p \div \{6^2 - (6^2 - 5^2)\cos^2\theta_p\}$

$\quad = 2.5^2\cos\theta_p \div 6\sin^2\psi_p.\quad r_c^2 = 6^2.5^2 \div \{6^2 - (6^2 - 5^2)\cos^2\theta_c\},$

$r_c = 5 \div \sin\psi_c.\quad r_f = 5^2 \div \{6 + \sqrt{6^2 - 5^2}.\cos\theta_f\} = 5^2 \div 2.6\cos^2\dfrac{\psi_f}{2}.$
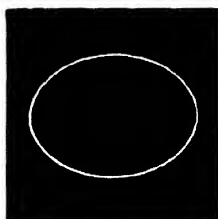
Fig. 7.   Axes as $5:4$.

Radii as $9:1$; $e=0\dot{.}1$.

$a=1$, $p=0{\cdot}8$; $5^2 e^2 = 5^2 - 4^2 = 9$.

$5^2 y^2 \div 4^2 = 2.5$ $x_e - x_e^2 = 5^2 - x_e^2$
$$= 4^2 + 2\sqrt{5^2-4^2}.x_f - x_f^2.$$

$r_e = 2.5.4^2\cos\theta_e \div \{5^2 - (5^2-4^2)\cos^2\theta_e\}$

$= 2.4^2\cos\theta_e \div 5\sin\psi_e.$ $r_c^2 = 5^2.4^2 \div \{5^2 - (5^2-4^2)\cos^2\theta_c\}$,

$r_c = 4 \div \sin\psi_c.$ $r_f = 4^2 \div \{5(+5^2-4^2)\cos\theta_f\} = 4^2 \div 2.5\cos^2\dfrac{\psi_f}{2}$
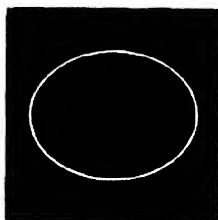
Fig. 8.   Axes as $9:7$.

Radii as $8:1$; $e=0{\cdot}125$.

$a=1$, $p=0\dot{.}7$; $9^2 e^2 = 9^2 - 7^2 = 32$.

$9^2 y^2 \div 7^2 = 2.9$ $x_e - x_e^2 = 9^2 - x_e^2$
$$= 7^2 + 2\sqrt{9^2-7^2}\,x_f - x_f^2,$$

$r_e = 2.9.7^2\cos\theta_e \div \{9^2 - (9^2-7^2)\cos^2\theta_e\}$

$= 2.7^2\cos\theta_e \div 9\sin^2\psi_e.$ $r_c^2 = 9^2.7^2 \div \{9^2 - (9^2-7^2)\cos^2\theta_c\}$,

$r_c = 7 \div \sin\psi_c.$ $r_f = 7^2 \div \{9 + \sqrt{9^2-7^2}.\cos\theta_f\} = 7^2 \div 2.9\cos^2\dfrac{\psi_f}{2}$

Fig. 9.   Axes as $4:3$.

Radii as $7:1$; $e=0\dot{.}142857\dot{.}$

$a=1$, $p=0{\cdot}75$; $4^2 e^2 = 4^2 - 3^2 = 7$.

$4^2 y^2 \div 3^2 = 2.4$ $x_e - x_e^2 = 4^2 - x_e^2$
$$= 3^2 + 2\sqrt{4^2-3^2}.x_f - x_f^2.$$

$r_e = 2.4.3^2\cos\theta_e \div \{4^2 - (4^2-3^2)\cos^2\theta_e\}$

$= 2.3^2\cos\theta_e \div 4\sin^2\psi_e.$ $r_c^2 = 4^2.3^2 \div \{4^2 - (4^2-3^2)\cos^2\theta_c\}$,

$r_c = 3 \div \sin\psi.$ $r_f = 3^2 \div \{4 + \sqrt{4^2-3^2}.\cos\theta_f\} = 3^2 \div 2.4\cos^2\dfrac{\psi_f}{2}.$

Fig. 10.   Axes as $7:5$.
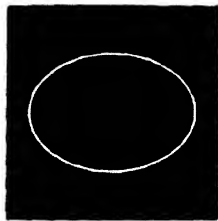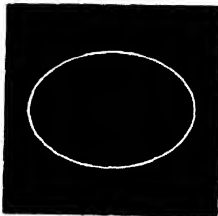
Radii as $6:1$; $e=0\cdot 16$.

$a=1,\ p=0\cdot 714285$; $7^2e^2=7^2-5^2=24$.

$7^2y^2\div 5^2=2.7x_v-x_v^2=7^2-x_e^2$

$\qquad\qquad =5^2+2\sqrt{7^2-5^2}.x_f-x_f^2$.

$r_v=2.7.5^2\cos\theta_v\div\{7^2-(7^2-5^2)\cos^2\theta_v\}$

$=2.5^2\cos\theta_v\div 7\sin^2\psi_v$. $r_c^2=7^2.5^2\div\{7^2-(7^2-5^2)\cos^2\theta_c\}$,

$r_e=5\div\sin\psi_e$. $r_f=5^2\div\{7+\sqrt{7^2-5^2}.\cos\theta_f\}=5^2\div 2.7\cos^2\dfrac{\psi_f}{2}$.

Fig. 11.   Axes as $10:7$.

Radii as $17:3$; $e=0\cdot 17647$.

$a=1,\ p=0\cdot 7$; $10^2e^2=10^2-7^2=51$.

$10^2y^2\div 7^2=2.10x_v-x_v^2=10^2-x_e^2$

$\qquad\qquad =7^2+2\sqrt{10^2-7^2}.x_f-x_f^2$.

$r_v=2\ 10.7^2\cos\theta_v\div\{10^2-(10^2-7^2)\cos^2\theta_v\}$

$=2.7^2\cos\theta_v\div 10\sin^2\psi_v$. $r_c^2=10^2.7^2\div\{10^2-(10^2-7^2)\cos^2\theta_c\}$,

$r_e=7\div\sin\psi_e$. $r_f=7^2\div\{10+\sqrt{10^2-7^2}.\cos\theta_f\}=7^2-2.10\cos^2\dfrac{\psi_f}{2}$.
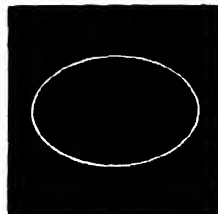
Fig. 12.   Axes as $3:2$.

Radii as $5:1$; $e=0\cdot 2$.

$a=1,\ p=0\cdot 6$; $3^2e^2=3^2-2^2=5$.

$3^2y^2\div 2^2=2.3x_v-x_v^2=3^2-x_e^2$

$\qquad\qquad =2^2+2\sqrt{3^2-2^2}.x_f-x_f^2$.

$r_v=2.3.2^2\cos\theta_v\div\{3^2-(3^2-2^2)\cos^2\theta_v\}$

$=2.2^2\cos\theta_v\div 3\sin^2\psi_v$. $r_c^2=3^2.2^2\div\{3^2-(3^2-2^2)\cos^2\theta_c\}$,

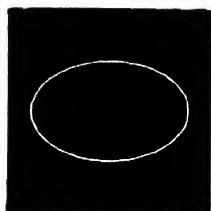$r_e=2\div\sin\psi_e$. $r_f=2^2\div\{3+\sqrt{3^2-2^2}.\cos\theta_f\}=2^2\div 2.3\cos^2\dfrac{\psi_f}{2}$.

Fig. 13.   Axes as 8 : 5.

Radii as 13 : 3; $e = 0.23$.

$a = 1, p = 0.625$; $8^2 e^2 = 8^2 - 5^2 = 39.$

$8^2 y^2 \div 5^2 = 2.8 x_v - x_v^2 = 8^2 - x_c^2$

$$= 5^2 + 2 \sqrt{8^2 - 5^2}.x_f - x_f^2.$$

$r_v = 2.8.5^2 \cos \theta_v \div \{8^2 - (8^2 - 5^2) \cos^2 \theta_v\}$

$= 2.5^2 \cos \theta_v \div 8 \sin^2 \psi_v.$ $r_c^2 = 8^2.5^2 \div \{8^2 - (8^2 - 5^2) \cos^2 \theta_c\},$

$r_e = 5 \div \sin \psi_e.$ $r_f = 5^2 \div \{8 + \sqrt{8^2 - 5^2}.\cos \theta_f\} = 5^2 \div 2.8 \cos^2 \dfrac{\psi_f}{2}.$
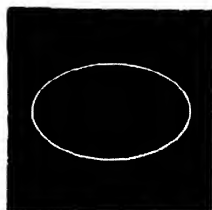

Fig. 14.   Axes as 5 : 3.

Radii as 4 : 1; $e = 0.25$.

$a = 1, p = 0.6$; $5^2 e^2 = 5^2 - 3^2 = 16.$

$5^2 y^2 \div 3^2 = 2.5 x_v - x_v^2 = 5^2 - x_c^2$

$$= 3^2 + 2 \sqrt{5^2 - 3^2}.x_f - x_f^2.$$

$r_v = 2.5.3^2 \cos \theta_v \div \{5^2 - (5^2 - 3^2) \cos^2 \theta_v\}$

$= 2.3^2 \cos \theta_v \div 5 \sin^2 \psi_v.$ $r_c^2 = 5^2.3^2 \div \{5^2 - (5^2 - 3^2) \cos^2 \theta_c\},$

$r_e = 3 \div \sin \psi_e.$ $r_f = 3^2 \div \{5 + \sqrt{5^2 - 3^2}.\cos \theta_f\} = 3^2 \div 2.5 \cos^2 \dfrac{\psi_f}{2}.$
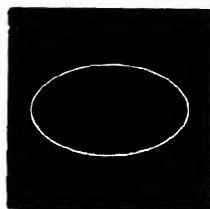

Fig. 15.   Axes as 7 : 4.

Radii as 11 : 3; $e = 0.27$.

$a = 1, p = 0.571428$; $7^2 e^2 = 7^2 - 4^2 = 33.$

$7^2 y^2 \div 4^2 = 2.7 x_v - x_v^2 = 7^2 - x_c^2$

$$= 4^2 + 2 \sqrt{7^2 - 4^2}.x_f - x_f^2.$$

$r_v = 2.7.4^2 \cos \theta_v \div \{7^2 - (7^2 - 4^2) \cos^2 \theta_v\}$

$= 2.4^2 \cos \theta_v \div 7 \sin^2 \psi_v.$ $r_c^2 = 7^2.4^2 \div \{7^2 - (7^2 - 4^2) \cos^2 \theta_c\},$

$r_c = 4 \div \sin \psi_c.$ $r_f = 4^2 \div \{7 + \sqrt{7^2 - 4^2}.\cos \theta_f\} = 4^2 \div 2.7 \cos^2 \dfrac{\psi_f}{2}.$

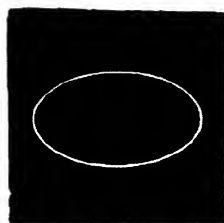**Fig. 16.   Axes as 9 : 5.**

Radii as 7 : 2; $e = 0.\overset{.}{2}8571\overset{.}{4}$.

$a = 1, p = 0.\overset{.}{5}$; $9^2 s^2 = 9^2 - 5^2 = 56$.

$9^2 y^2 \div 5^2 = 2.9 x_v - x_v^2 = 9^2 - x_c^2$

$\qquad = 5^2 + 2\sqrt{9^2 - 5^2}.x_f - x_f^2.$

$r_v = 2.9.5^2 \cos\theta_v \div \overline{9^2 - (9^2 - 5^2)\cos^2\theta_v\}}$

$\quad = 2.5^2 \cos\theta_v \div 9\sin^2\psi_v.$ $r_c^2 = 9^2.5^2 \div \{9^2 - (9^2 - 5^2)\cos^2\theta_c\},$

$r_c = 5 \div \sin\psi_c.$ $r_f = 5^2 \div \{9 + \sqrt{9^2 - 5^2}.\cos\theta_f\} = 5^2 \div 2.9\cos^2\dfrac{\psi_f}{2}.$
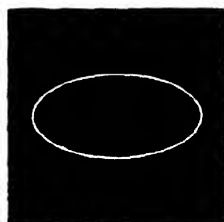

Fig. 17.   Axes as 2 : 1.

Radii as 3 : 1; $e = 0.\overset{.}{3}$.

$a = 1, p = 0.5$; $2^2 s^2 = 2^2 - 1^2 = 3$.

$2^2 y^2 \div 1^2 = 2.2 x_v - x_v^2 = 2^2 - x_c^2$

$\qquad = 1^2 + 2\sqrt{2^2 - 1^2}x_f - x_f^2.$

$r_v = 2.2.1^2 \cos\theta_v \div \overline{2^2 - (2^2 - 1^2)\cos^2\theta_v\}}$

$\quad = 2.1^2 \cos\theta_v \div 2\sin^2\psi_v.$ $r_c^2 = 2^2.1^2 \div \{2^2 - (2^2 - 1^2)\cos^2\theta_c\},$

$r_c = 1 \div \sin\psi_c.$ $r_f = 1^2 \div \{2 + \sqrt{2^2 - 1^2}.\cos\theta_f\} = 1^2 \div 2.2\cos^2\dfrac{\psi_f}{2}.$
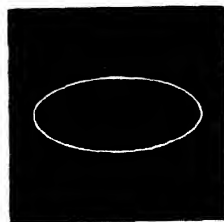

Fig. 18.   Axes as 9 : 4.

Radii as 13 : 5; $e = 0.3846$.

$a = 1, p = 0.\overset{.}{4}$; $9^2 s^2 = 9^2 - 4^2 = 65$.

$9^2 y^2 \div 4^2 = 2.9 x_v - x_v^2 = 9^2 - x_c^2$

$\qquad = 4^2 + 2\sqrt{9^2 - 4^2}.x_f - x_f^2.$

$r_v = 2.9.4^2 \cos\theta_v \div \overline{9^2 - (9^2 - 4^2)\cos^2\theta_v\}}$

$\quad = 2.4^2 \cos\theta_v \div 9\sin^2\psi_v.$ $r_c^2 = 9^2.4^2 \div \{9^2 - (9^2 - 4^2)\cos^2\theta_c\},$

$r_c = 4 \div \sin\psi_c.$ $r_f = 4^2 \div \{9 + \sqrt{9^2 - 4^2}.\cos\theta_f\} = 4^2 \div 2.9\cos^2\dfrac{\psi_f}{2}.$

Fig. 19. Axes as 7 : 3.

Radii as 5 : 2; $e = 0\cdot4$.

$a = 1$, $p = 0\cdot\dot{4}2857\dot{1}$; $7^2t^2 = 7^2 - 3^2 = 40$.

$7^2y^2 \div 3^2 = 2.7\,x_e - x_e^2 = 7^2 - x_e^2$

$\qquad = 3^2 + 2\sqrt{7^2 - 3^2}.x_f - x_f^2$

$r_e = 2.7.3^2 \cos\theta_e + \{7^2 - (7^2 - 3^2)\cos^2\theta_e\}$

$\quad = 2.3^2\cos\theta_e + 7\sin^2\psi_e.$ $\quad r_c^2 = 7^2.3^2 + \{7^2 - (7^2 - 3^2)\cos^2\theta_c\},$

$r_e = 3 + \sin\psi_e.$ $\quad r_f = 3^2 + \{7 + \sqrt{7^2 - 3^2}.\cos\theta_f\} = 3^2 + 2.7\cos^2\dfrac{\psi_f}{2}.$

Fig. 20. Axes as 5 : 2.

Radii as 7 : 3; $e = 0\cdot\dot{4}2857\dot{1}$.

$a = 1$, $p = 0\cdot4$; $5^2t^2 = 5^2 - 2^2 = 21$.

$5^2y^2 \div 2^2 = 2.5\,x_e - x_e^2 = 5^2 - x_e^2$

$\qquad = 2^2 + 2\sqrt{5^2 - 2^2}.x_f - x_f^2.$

$r_e = 2.5.2^2 \cos\theta_e + \{5^2 - (5^2 - 2^2)\cos^2\theta_e\}$

$\quad = 2.2^2\cos\theta_e + 5\sin^2\psi_e.$ $\quad r_c^2 = 5^2.2^3 + \{5^2 - (5^2 - 2^2)\cos^2\theta_c\},$

$r_e = 2 + \sin\psi_e.$ $\quad r_f = 2^2 + \{5^2 + (5^2 - 2^2)\cos\theta_f\} = 2^2 + 2.5\cos^2\dfrac{\psi_f}{2}.$

Fig. 21. Axes as 8 : 3.

Radii as 11 : 5; $e = 0\cdot45$.

$a = 1$, $p = 0\cdot375$; $8^2t^2 = 8^2 - 3^2 = 55$.

$8^2y^2 \div 3^2 = 2.8\,x_e - x_e^2 = 8^2 - x_e^2$

$\qquad = 3^2 + 2\sqrt{8^2 - 3^2}.x_f \quad x_f^2.$

$r_e = 2.8.3^2\cos\theta \div_e \{8^2 - (8^2 - 3^2)\cos^2\theta_e\}$

$\quad = 2.3^2\cos\theta_e + 8\sin^2\psi_e.$ $\quad r_c^2 = 8^2.3^2 + \{8^2 - (8^2 - 3^2)\cos^2\theta_c\},$

$r_e = 3 \div \sin\psi_e.$ $\quad r_f = 3^2 \div \{8 + \sqrt{8^2 - 3^2}.\cos\theta_f\} = 3^2 \div 2.8\cos^2\dfrac{\psi_f}{2}.$
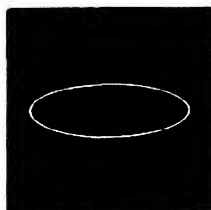
Fig. 22.   Axes as 3 : 1.

Radii as 2 : 1; $e = 0.5$.

$a = 1, p = 0.\dot{3}$; $3^2 e^2 = 3^2 - 1^2 = 8$.

$3^2 y^2 \div 1^2 = 2.3 x_v - x_v^2 = 3^2 - x_c^2$

$\qquad = 1^2 + 2\sqrt{3^2 - 1^2}.x_f - x_f^2$.

$r_v = 2.3.1^2\cos\theta_v \div \{3^2 - (3^2 - 1^2)\cos^2\theta_v\}$

$\quad = 2.1^2\cos\theta_v \div 3\sin^2\psi_v$. $r_c^2 = 3^2.1^2 \div \{3^2 - (3^2 - 1^2)\cos^2\theta_c\}$,

$r_c = 1 \div \sin\psi_c$. $r_f = 1^2 \div \{3 + \sqrt{3^2 - 1^2}\cos\theta_f\} = 1^2 \div 2.3\cos^2\dfrac{\psi_f}{2}$.

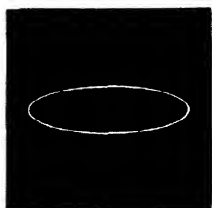**Fig. 23.   Axes as 10 : 3.**

Radii as 13 : 7; $e = 0.53846$.

$a = 1, p = 0.3$; $10^2 e^2 = 10^2 - 3^2 = 91$.

$10^2 y^2 \div 3^2 = 2.10 x_v - x_v^2 = 10^2 - x_c^2$

$\qquad = 3^2 + 2\sqrt{10^2 - 3^2}.x_f - x_f^2$.

$r_v = 2.10.3^2\cos\theta_v \div \{10^2 - (10^2 - 3^2)\cos^2\theta_v\}$

$\quad = 2.3^2\cos\theta_v \div 10\sin^2\psi_v$. $r_c^2 = 10^2.3^2 \div \{10^2 - (10^2 - 3^2)\cos^2\theta_c\}$,

$r_c = 3 \div \sin\psi_c$. $r_f = 3^2 \div \{10 + \sqrt{10^2 - 3^2}.\cos\theta_f\} = 3^2 \div 2.10\cos^2\dfrac{\psi_f}{2}$.

Fig. 24.   Axes as 7 : 2.

Radii as 9 : 5; $e = 0.5$.

$a = 1, p = 0.\dot{2}8571\dot{4}$; $7^2 e^2 = 7^2 - 2^2 = 45$.

$7^2 y^2 \div 2^2 = 2.7 x_v - x_v^2 = 7^2 - x_c^2$

$\qquad = 2^2 + 2\sqrt{7^2 - 2^2}.x_f - x_f^2$.

$r_v = 2.7.2^2\cos\theta_v \div \{7^2 - (7^2 - 2^2)\cos^2\theta_v\}$

$\quad = 2.2^2\cos\theta_v \div 7^2\sin^2\psi_v$. $r_c^2 = 7^2.2^2 \div \{7^2 - (7^2 - 2^2)\cos^2\theta_c\}$,

$r_c = 2 \div \sin\psi_c$. $r_f = 2^2 \div \{7 + \sqrt{7^2 - 2^2}.\cos\theta_f\} = 2^2 \div 2.10\cos^2\dfrac{\psi_f}{2}$.
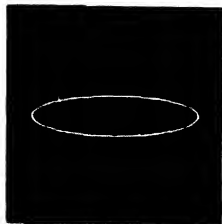
Fig. 25. Axes as 4 : 1.

Radii as 5 : 3 ; $e = 0\cdot6$.

$a = 1, p = 0\cdot25 ; 4^2 \iota^2 = 4^2 - 1^2 = 15.$

$4^2 y^2 \div 1^2 = 2.4\, x_v - x_v^2 = 4^2 - x_c^2$

$\qquad = 1^2 + 2\sqrt{4^2 - 1^2}.\, x_f - x_f^2.$

$r_v = 2.4.1^2 \cos\theta_v \div \{4^2 - (4^2 - 1^2)\cos^2\theta_v\}$

$\quad = 2.1^2 \cos\theta_v + 4\sin^2\psi_v.\quad r_c^2 = 4^2.1^2 + \{4^2 - (4^2 - 1^2)\cos^2\theta_c\}$

$r_c = 1 + \sin\psi_v.\quad r_f = 1^2 + \{4 + \sqrt{4^2 - 1^2}.\cos\theta_f\} = 1^2 + 2.4\cos^2\dfrac{\psi_f}{2}.$

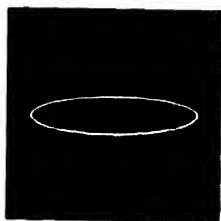Fig. 26. Axes as 9 : 2.

Radii as 11 : 7 ; $e = 0\cdot\dot{6}\dot{3}$.

$a = 1,\ p = 0\cdot2 ;\ 9^2\iota^2 = 9^2 - 2^2 = 77.$

$9^2 y^2 \div 2^2 = 2.9 x_v - x_v^2 = 9^2 - x_c^2$

$\qquad = 2^2 + 2\sqrt{9^2 - 2^2}.x_f - x_f^2.$

$r_v = 2.9.2^2 \cos\theta_v + \{9^2 - (9^2 - 2^2)\cos^2\theta_v\}$

$\quad = 2.2^2 \cos\theta_v + 9\sin^2\psi_v.\quad r_c^2 = 9^2.2^2 + \{9^2 - (9^2 - 2^2)\cos^2\theta_c\},$

$r_c = 2 + \sin\psi_v.\quad r_f = 2^2 + \{9 + \sqrt{9^2 - 2^2}.\cos\theta_f\} = 2^2 + 2.9\cos^2\dfrac{\psi_f}{2}.$

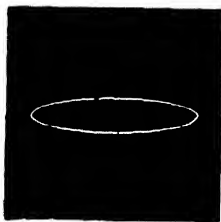Fig. 27. Axes as 5 : 1.

Radii as 3 : 2 ; $e = 0\cdot\dot{6}$.

$a = 1,\ p = 0\cdot2 ;\ 5^2\iota^2 = 5^2 - 1^2 = 24.$

$5^2 y^2 + 1^2 = 2.5 x_v - x_v^2 = 5^2 - x_c^2$

$\qquad = 1^2 + 2\sqrt{5^2 - 1^2}.x_f - x_f^2.$

$r_v = 2.5.1\ \cos\theta_v \div \{5^2 - (5^2 - 1^2)\cos^2\theta_v\}$

$\quad = 2.1^2 \cos\theta_v + 5\sin^2\psi_v.\quad r_c^2 = 5^2.1^2 + \{5^2 - (5^2 - 1^2)\cos^2\theta_c\},$

$r_c = 1 \div \sin\psi_c.\quad r_f = 1^2 \div \{5 + \sqrt{5^2 - 1^2}.\cos\theta_f\} = 1^2 \div 2.5\cos^2\dfrac{\psi_f}{2}.$
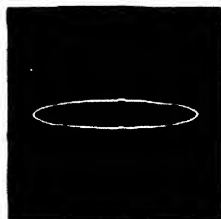
Fig. 28. Axes as 6 : 1.

Radii as $7 : 5$; $e = 0\cdot\dot{7}14285$.

$a = 1$, $p = 0\cdot\dot{1}6$; $6^2 \cdot 1^2 = 6^2 - 1^2 = 35$

$6^2 y^2 \div 1^2 = 2 . 6\, x_v - x_v^2 = 6^2 - x_c^2$

$$= 1^2 + 2\sqrt{6^2 - 1^2}\, . x_f - x_f^2.$$

$r_v = 2 . 6 . 1^2 \cos\theta_v \div \; 6^2 - (6^2 - 1^2)\cos^2\theta_v$

$= 2 . 1^2 \cos\theta_v \div 6\sin^2\psi_v.$ $r_c^2 = 6^2 . 1^2 \div \{6^2 - (6^2 - 1^2)\cos^2\theta_c\}$,

$r_c = 1 \div \sin\psi_c.$ $r_f = 1^2 \div \{6 + \sqrt{6^2 - 1^2}.\cos\theta_f\} = 1^2 \div 2 . 6\cos^2\dfrac{\psi_f}{2}.$

Fig. 29. Axes as 7 : 1.

Radii as $4 : 3$; $e = 0\cdot75$.

$a = 1$, $p = 0\cdot\dot{1}42857$; $7^2 \cdot 1^2 = 7^2 - 1^2 = 48$.

$7^2 y^2 \div 1^2 = 2 . 7\, x_v - x_v^2 = 7^2 - x_c^2$

$$= 1^2 + 2\sqrt{7^2 - 1^2}\, x_f - x_f^2.$$

$r_v = 2 . 7 . 1^2 \cos\theta_v \div \; 7^2 - (7^2 - 1^2)\cos^2\theta_v\}$

$= 2 . 1^2 \cos\theta_v \div 7\sin^2\psi_v.$ $r_c^2 = 7^2 . 1^2 \div \{7^2 - (7^2 - 1^2)\cos^2\theta_c\}$,

$r_c = 1 \div \sin\psi_c.$ $r_f = 1^2 \div \{7 + \sqrt{7^2 - 1^2}.\cos\theta_f\} = 1^2 \div 2 . 7\cos^2\dfrac{\psi_f}{2}.$

Fig. 30. Axes as 8 : 1.

Radii as $9 : 7$; $e = 0\cdot\dot{7}$.

$a = 1$, $p = 0\cdot125$; $8^2 \cdot 1^2 = 8^2 - 1^2 = 63$.

$8^2 y^2 \div 1^2 = 2 . 8\, x_v - x_v^2 = 8^2 - x_c^2$

$$= 1^2 + 2\sqrt{8^2 - 1^2}\, x_f - x_f^2.$$

$r_v = 2 . 8 . 1^2 \cos\theta_v \div \{8^2 - (8^2 - 1^2)\cos^2\theta_v\}$

$= 2 . 1^2 \cos\theta_v \div 8\sin^2\psi_v.$ $r_c^2 = 8^2 . 1^2 \div \{8^2 - (8^2 - 1^2)\cos^2\theta_c\}$,

$r_c = 1 \div \sin\psi_c.$ $r_f = 1^2 \div \{8 + \sqrt{8^2 - 1^2}.\cos\theta_f\} = 1^2 \div 2 . 8\cos^2\dfrac{\psi_f}{2}.$
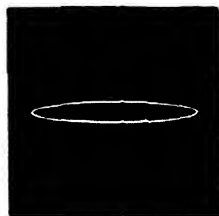
Fig. 31. Axes as 9 : 1.

Radii as 5 : 4; $e = 0.8$.

$u = 1$, $p = 0.1$; $9^2 e^2 = 9^2 - 1^2 = 80$

$9^2 y^2 \div 1^2 = 2.9 \, x_o - x_o^2 = 9^2 - x_c^2$

$= 1^2 + 2 \sqrt{9^2 - 1^2} . x_f - x_f^2.$

$r_o = 2.9.1^2 \cos \theta_o \div \{9^2 - (9^2 - 1^2) \cos^2 \theta_o\}$

$= 2.1^2 \cos \theta_o \div 9 \sin^2 \psi_o.$ $r_c^2 = 9^2.1^2 \div \{9^2 - (9^2 - 1^2) \cos^2 \theta_c\},$

$r_c = 1 \div \sin \psi_c.$ $r_f = 1^2 \div \{9 + \sqrt{9^2 - 1^2} . \cos \theta_f\} = 1^2 \div 2.9 \cos^2 \dfrac{\psi_f}{2}.$

Fig. 32. Axes as 10 : 1.

Radii as 11 : 9; $e = 0.81$.

$u = 1$, $p = 0.1$; $10^2 e^2 = 10^2 - 1^2 = 99.$

$10^2 y^2 \div 1^2 = 2.10 \, x_o - x_o^2 = 10^2 - x_c^2$

$= 1^2 + 2 \sqrt{10^2 - 1^2} . x_f - x_f^2.$

$r_o = 2.10.1^2 \cos \theta_o \div \{10^2 - (10^2 - 1^2) \cos^2 \theta_o\}$

$= 2.1^2 \cos \theta_o \div 10 \sin^2 \psi_o.$ $r_c^2 = 10^2.1^2 \div \{10^2 - (10^2 - 1^2) \cos^2 \theta_c\},$

$r_c = 1 \div \sin \psi_c.$ $r_f = 1^2 \div \{10 + \sqrt{10^2 - 1^2} . \cos \theta_f\} = 1^2 \div 2.10 \cos^2 \dfrac{\psi_f}{2}.$

Fig. 33. Axes as 1 : 0.

Radii as 1 : 1; $e = 1$.

$u = 1$, $p = 0$; $1^2 e^2 = 1^2 - 0^2 = 1$ or $e = 1$.

The straight line, the other extreme of the Ellipse, see fig. 1.

Foci at the vertices. $y = 0$.

$r_o = r_f = x_o = x_f.$ $r_c = x_c.$

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October 1855.